Trend and Mean Reversion Modelling in a Market with Heterogeneous Investors: A Dynamical Systems Approach

Master Thesis Presentation

by Seth Tolev
Supervisors: Prof. Dr. Walter Farkas, Dr. Matthias Feiler

ETH Zurich / University of Zurich

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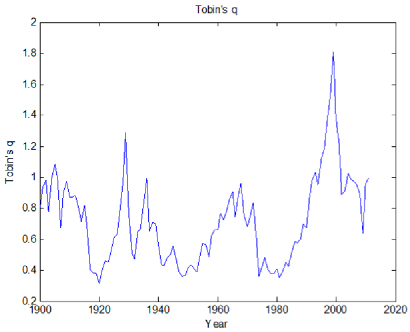
Outline

Overview

- Tobin's q
- Momentum and Mean Reversion
- Nonlinear Dynamical Theory

Tobin's q

- The ratio between the value of companies according to the stock market and their net worth measured at replacement cost
- A commonly used measure of the value of the stock market
- Calculated using the Flow of Funds Accounts of the United States Z1, published by the Federal Reserve

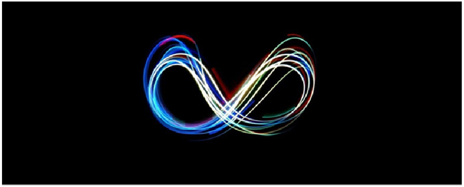


Momentum and Mean Reversion

- Momentum is the empirically observed tendency for rising asset prices to rise further, and falling prices to keep falling
- Mean reversion is a theory suggesting that prices and returns eventually move back towards the mean or average (e.g. historical price/return average, average growth in the economy, or the average return of an industry)
- The simultaneous existence of these phenomena has been a puzzle, but some researchers (e.g. Hong and Stein 1999) have argued that it could be explained by investor heterogeneity

Nonlinear Dynamical Theory

- Dynamical theory is an area of mathematics used to describe the behaviour of complex dynamical systems by employing differential or difference equations
- A number of applications in physics, biology, meteorology, medicine, and chemistry
- Example: Lorenz System



Source: University of Waterloo

Overview

- IS-LM Model
- Blanchard Model
- Boundedly Rational Heterogeneous Agents Model

IS-LM Model

- Depicts relationship between real income and interest rates in the goods and money markets

$$\dot{y} = \beta_y(e - y) = \beta_y \left[C(y) + I(i, y) + g - y \right] \quad (1)$$

$$\dot{i} = \beta_i(L(i, y) - m_0) \quad (2)$$

y – real income
e – planned spending
i – interest rate
C – consumption function
I – investment function
g – government spending
m_0 – monetary supply
$L(i,y)$ – demand for real money balances

Blanchard Model (1981)

- Extends the IS-LM model by assuming that in the IS equation investment demand depends on Tobin's q (market valuations feed back into the real sector)

$$\dot{y} = \beta_y(\alpha_0 + \alpha_1 y + \alpha_2 q) \tag{3}$$

$$i = h_0 + h_1 y \tag{4}$$

$$\frac{\dot{q}}{q} + \frac{Py - \delta}{q} = i \tag{5}$$

$\alpha_0 > 0$, $\alpha_1 \in$[-1,0), $\alpha_2 > 0$, $h_0 < 0$, $h_1 > 0$, $P > 0$ and $\delta > 0$

- Money market interest rate adjusts instantaneously to LM equilibrium
- Perfect substitution - expected rate of return for stocks is the interest rate in the short term
- Zero capital gains and real profits equal to the interest rate in equilibrium: real profits affect stock market value
- Perfect foresight - no distinction between expected and actual returns

Boundedly Rational Heterogeneous Agents Model (2006)

- An extension of the Blanchard Model
- Less than infinite adjustment speeds in market valuations equation (imperfect asset substitution), regime switching
- Two groups of investors are introduced: fundamentalists and chartists, controlled by proportion parameters n^c and n^f, where $n^c + n^f = 1$
- Imperfect foresight i.e. asset valuations q and their expectations by the two groups of investors (z^f, z^c) follow different dynamics

Boundedly Rational Heterogeneous Agents Model (2006)

- The model in full form is rather messy

$$\dot{y} = \beta_y(e - y) = \beta_y(C + c(y - \delta K - T) + I + \psi(q - 1) + \delta K + g - y) \quad (6)$$

$$dq = (n^f \beta_q^f \epsilon^f + n^c \beta_q^c \epsilon^c) q dt + \sigma q dW \quad (7)$$

$$\epsilon^x = \frac{\frac{(1-v)y}{K} - \delta}{q} + z^x - i, \; x \in (c, f) \quad (8)$$

$$i = h_0 + h_1 y \quad (9)$$

$$\dot{z}^c = \beta_{z^c}\left(\frac{\dot{q}}{q} - z^c\right) \quad (10)$$

$$z^f = E\left[\frac{\dot{q}}{q}\right] = n^f \beta_q^f \epsilon^f + n^c \beta_q^c \epsilon^c \quad (11)$$

- Can obtain Blanchard Model as a limiting case by setting $n^c = 0$, $\sigma = 0$, $\beta_q^f = \beta_q^c = \beta_{z^c} = \infty$

Boundedly Rational Heterogeneous Agents Model (2006) - Simplified

Setting $\alpha_0 = \beta_y(C + I + g - \psi + \delta K)$, $\alpha_1 = \beta_y(c - 1)$, $\alpha_2 = \beta_y \psi$, $P = \frac{(1-v)}{K}$, and $n^f = 1 - n^c$and assuming $\sigma = 0$, $\beta_q^f = \beta_q^c$, and no regime switching, we get a non-linear dynamical system of the following form:

$$\dot{y} = \alpha_0 + \alpha_1 y + \alpha_2 q \tag{12}$$

$$\dot{q} = \frac{\beta_q}{1 - \beta_q + \beta_q n^c}\left(-(h_0 + h_1 y)q + n^c q z^c + Py - \delta\right) \tag{13}$$

$$\dot{z}^c = \frac{\beta_q \beta_{z^c}}{1 - \beta_q + \beta_q n^c}\left((1 - \frac{1}{\beta_q})z^c + \frac{Py - \delta}{q} - h_0 - h_1 y\right) \tag{14}$$

where $\alpha_1 \in [-1, 0)$, $\alpha_2 > 0$, $h_0 < 0$, $h_1 > 0$, $P > 0$, $\beta_q > 0$, $n^c \in [0, 1]$, $\beta_{z^c} > 0$, $P > 0$, $\delta > 0$

Overview

- Goals:
  - Reduce dimensionality in order to obtain an analytically tractable problem
  - Derive stability and system bifurcation conditions
  - Understand model system dynamics under different parameterizations
  - Identify system parameter ranges that result in economically interesting behaviour
  - Provide an economic interpretation of amplitude of market cycles in the context of the model

Dimension Reduction and Steady States

1. Removing the dependence on z^c (set $n^c = 0$), we obtain the "Real Economy Subsystem"

$$\dot{y} = \alpha_0 + \alpha_1 y + \alpha_2 q \tag{15}$$

$$\dot{q} = \frac{\beta_q}{1-\beta_q}\left[-(h_0 + h_1 y)q + Py - \delta\right] \tag{16}$$

2. Assuming fixed $y = \bar{y}$ (output), we obtain the "Financial Market Subsystem", $\Pi = P\bar{y} - \delta$, $i = h_0 + h_1\bar{y}$

$$\dot{q} = \frac{\beta_q}{1-\beta_q+\beta_q n^c}\left[-iq + n^c q z^c + \Pi\right] \tag{17}$$

$$\dot{z}^c = \frac{\beta_q \beta_{z^c}}{1-\beta_q+\beta_q n^c}\left[(1-\frac{1}{\beta_q})z^c + \frac{\Pi}{q} - i\right] \tag{18}$$

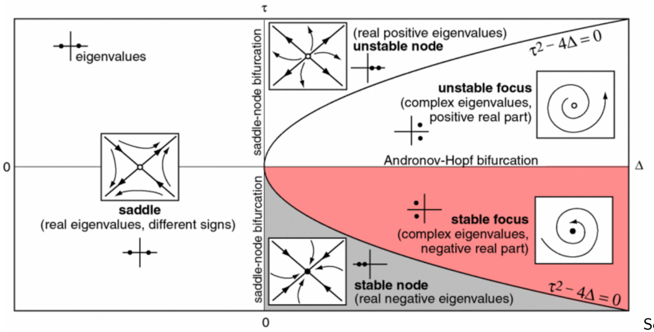
Steady States:

$$q = \frac{\Pi}{i}, z^c = 0 \tag{19}$$

Stability Analysis

- It can be shown that both subsystems of the model satisfy the property of "almost linearity", hence their hyperbolic steady states can be classified using linear systems analysis
- The derived linear systems may provide an approximation to "almost linear" systems near a hyperbolic equilibrium point, however, globally their trajectories may differ significantly!
- In practice, this means that an equilibrium type can be determined by examining the eigenvalues of the system's Jacobian matrix
- The resulting complex algebraic expressions can be simplified by using a known relationship between a matrix's eigenvalues and its trace τ and determinant Δ functions

Stability Analysis



Source:Scholarpedia

Caveat: In two dimensional almost linear systems, when the Jacobian matrix has two equal eigenvalues or when both eigenvalues have zero real part, the nonlinearities of the system can force spiralling behaviour (non-hyperbolic equilibria)

Derivation of Stability/Instability Conditions

Submodel 1: Real Economy

1. Find Jacobian Matrix

$$J = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \frac{\beta_q}{1-\beta_q}(-h_1\bar{q} + P) & \frac{\beta_q}{1-\beta_q}(-h_0 - h_1\bar{y}) \end{bmatrix}$$

2. Compute Trace and Determinant

$$\tau = \alpha_1 + \frac{\beta_q}{1-\beta_q}(-h_0 - h_1\bar{y}),\ \Delta = \frac{\beta_q}{1-\beta_q}\left[\alpha_1(h_0 + h_1\bar{y}) - \alpha_2(-h_1\bar{q} + P)\right]$$

3. Solve Inequality system $[\Delta > 0,\ \tau < 0]$ to obtain local stability condition

Proposition: Suppose that the market speed of adjustment β_q and the investment parameter α_2 are not zero, and that the equilibrium interest rate is positive. The equilibrium point $(\bar{y}, \bar{q})$ of the real sector subsystem (15-16) is locally stable when market adjustment speed is low $(\beta_q < 1)$ and $\frac{\alpha_2(h_1\bar{q} - P)}{h_0 + h_1 y} < \alpha_1 < \frac{\beta_q}{1-\beta_q}(h_0 + h_1\bar{y})$.

Derivation of Stability/Instability Conditions

Submodel 2: Financial Market

1. Find Jacobian Matrix and 2. Compute Trace and Determinant

$$J = \begin{bmatrix} \frac{-\beta_q i}{1-\beta_q+\beta_q n^c} & \frac{\beta_q \Pi n^c}{(1-\beta_q+\beta_q n^c)i} \\ \frac{-\beta_{z^c}\beta_q i^2}{(1-\beta_q+\beta_q n^c)\Pi} & \frac{-\beta_{z^c}\beta_q(1-\frac{1}{\beta_q})}{1-\beta_q+\beta_q n^c} \end{bmatrix}, \ \tau = \frac{-i\beta_q-\beta_q\beta_{z^c}+\beta_{z^c}}{1-\beta_q+\beta_q n^c}, \ \Delta = \frac{\beta_q i \beta_{z^c}}{1-\beta_q+\beta_q n^c}$$

3. Solve Inequalities [$\Delta > 0$, $\tau < 0$] and $\tau^2 - 4\Delta < 0$ respectively to obtain local stabiltiy/oscillation conditions

Proposition: *The equilibrium point $(\bar{q}, \bar{z}^c)$ of the financial market subsystem, described by Eqs.* (17) *and* (18), *is locally stable when one of the following cases is true:*

(i) $i < 0$ *and* $\beta_q < \frac{1}{1-n^c}$.

(ii) $i > 0$ *and* $\beta_q > \frac{1}{1-n^c}$.

(iii) $i > \frac{\beta_{z^c}(\beta_q-1)}{\beta_q}$ *and* $\frac{1}{1-n^c} > \beta_q > 1$

Proposition: *Assume positive interest rates and presence of chartists in the market* ($n^c > 0$). *The equilibrium point* $(\bar{q}, \bar{z}^c)$ *of the financial market subsystem, described by Eqs.* (17) *and* (18), *exhibits local oscillatory behaviour when*

$$\frac{\beta_{z^c} + i + 2\sqrt{i\beta_{z^c} n^c}}{i^2 + 2i\beta_{z^c} + \beta_{z^c}^2 - 4i\beta_{z^c} n^c \beta_{z^c}} < \beta_q < \frac{\beta_{z^c} + i - 2\sqrt{i\beta_{z^c} n^c}}{i^2 + 2i\beta_{z^c} + \beta_{z^c}^2 - 4i\beta_{z^c} n^c \beta_{z^c}} \quad (20)$$

Bifurcation Analysis

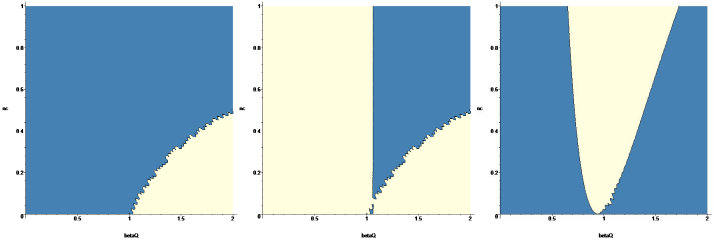
- Bifurcation theory is the study of changes in the qualitative structure of a system trajectories as parameters are varied
- A parameter value for which the trajectories do not have stable orbit structure is called a bifurcation value
- Knowledge of the bifurcation values is necessary for a complete understanding of the system

Theorem

Hopf Bifurcation

Let $\dot{x} = f(x, \mu)$ have a fixed point $(\bar{x}_0, \mu_0)$ at which (1) The Jacobian at $(\bar{x}_0, \mu_0)$ has a pair of purely imaginary eigenvalues and no other eigenvalues with real zero parts. This implies that there is a smooth curve of fixed points $(\bar{x}(\mu),\mu)$ with $\bar{x}(\mu_0) = \bar{x}_0$. If moreover, (2) $\frac{d(Re\lambda(\mu)}{d\mu}_{|\mu=\mu_0} > 0$, then there exist some periodic solutions bifurcating from $\bar{x}(\mu_0)$ and the period of the solutions is close to $\frac{2\pi}{\beta_0}$ where $\beta_0 = \frac{\lambda(\mu_0)}{i}$.

Bifurcation Analysis

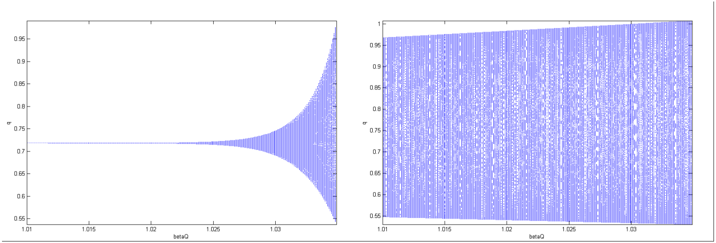


Determinant and trace boundary maps in n^c, β_q space: Δ (left), τ (center) and $\tau^2 - 4\Delta$ (right) for $\Pi = 0.0176$, $i = 0.0245, \beta_{zc} = 0.43$, blue is positive, yellow is negative

- Bifurcation boundaries can be represented as plots of the trace and determinant functions set to zero in parameter space
- The Hopf bifurcation region corresponds to a situation where the determinant is positive and the trace is zero

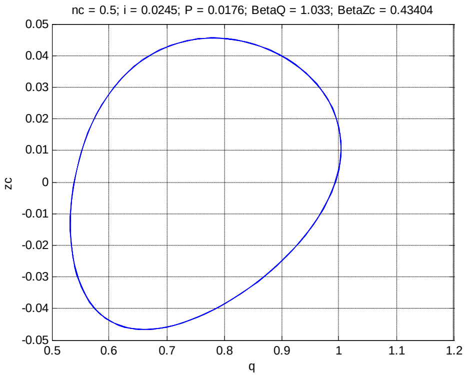
Bifurcation Analysis

- At $\beta_q = \frac{\beta_{z^c}}{\beta_{z^c - i}}$, the trace function has a zero crossing, and so a Hopf bifurcation is expected (the determinant is positive in this case)
- By the Hopf Bifurcation Theorem, since both eigenvalues are pure imaginary numbers, and $\frac{d(Re\lambda(\beta_q))}{d\beta_q} = \frac{d(-\tau(\beta_q))}{d\beta_q} = \frac{i+\beta_{z^c} n^c}{(1-\beta_q+\beta_q n^c)^2} > 0$ for all β_q, there exists a periodic solution



Left: $\beta_{z^c} = 0.42$, **Right:** $\beta_{z^c} = \frac{i\beta_q}{\beta_{q-1}}$,

Bifurcation Analysis



The system exhibits constant oscillations!

Linearization and Analytical Solution in Center Dynamics Case

- The linearization of the system can be expressed as:

$$\begin{pmatrix} \dot{q}_L \\ \dot{z}_L^c \end{pmatrix} = \begin{bmatrix} \frac{-\beta_q i}{1-\beta_q+\beta_q n^c} & \frac{\beta_q \Pi n^c}{(1-\beta_q+\beta_q n^c) i} \\ \frac{-\beta_{z^c} \beta_q i^2}{(1-\beta_q+\beta_q n^c)\Pi} & \frac{-\beta_{z^c} \beta_q (1-\beta_q)}{1-\beta_q+\beta_q n^c} \end{bmatrix} \begin{pmatrix} q_L \\ z_L^c \end{pmatrix} + \begin{pmatrix} \frac{\beta_q}{1-\beta_q+\beta_q n^c}(\Pi - n^c \bar{z}^c \bar{q}) \\ \frac{\beta_q \beta_{z^c}}{1-\beta_q+\beta_q n^c}(\frac{2\Pi}{\bar{q}} - i) \end{pmatrix} \quad (21)$$

- Expressing this as a second order ODE, and setting the constant term to zero, we get an equation of the form: $\ddot{q}_L - Tr(J)\dot{q}_L + Det(J)q_L = 0$
- Note that when $\text{Tr}(J) = 0$, i.e. the system undergoes a Hopf bifurcation, the equation reduces further to the equation of the harmonic oscillator from classical mechanics: $\ddot{q}_L + \omega^2 q_L = 0$, where $\omega = \sqrt{Det(A)}$ is the frequency of the oscillation
- The solution of the form $q_L(t) = A\cos(\omega t + \varphi)$ can be found, where A represents the amplitude of the oscillation

Linearization and Analytical Solution in Center Dynamics Case

- The solution can be found to be

$$q_L(t) = \sqrt{q_L(0)^2 + \frac{\beta_q(-iq_L(0) + n^c \bar{q} z^c(0))^2}{(1 - \beta_q + \beta_q n^c) i \beta_{z^c}}} \times$$
$$\cos\left[\sqrt{\frac{\beta_q i \beta_{z^c}}{1 - \beta_q + \beta_q n^c}} t + \arctan\left(\frac{\beta_q(-iq_L(0) + n^c \bar{q} z^c(0))}{q_L(0)(1 - \beta_q + \beta_q n^c)\sqrt{\frac{\beta_q i \beta_{z^c}}{1 - \beta_q + \beta_q n^c}}}\right)\right] \quad (22)$$

- When the linearized model exhibits stable oscillations, the amplitude increases with the initial condition and n^c
- In other words, oscillations in market valuations in the short term are amplified by the current valuation and the presence of chartists in the market

Overview

- Data
- Methodology
- Results

Data and Methodology

- US quarterly data on Tobin's q from 1952 to 2011, from the Federal Reserve
- z^c series: derived from the order of a pair of 4 quarter and 2 quarter moving averages on q (very common momentum strategy employed by chartists)
- System Identification of the $y - q$ dynamical system, using z^c as an external signal; note that lagged variables are not used as regressors
- Best fit least squares criterion computed as follows:

$$Best\ Fit = \left(1 - \frac{|y - \hat{y}|}{|y - \bar{y}|}\right) \times 100 \tag{23}$$

- Cross-Validation: technique used to assess how accurately a model would perform out-of-sample, best practice to avoid over-fitting the data
  - Use of separate estimation and validation datasets (first 150 data points and last 89 data points respectively)
- Residual Analysis

Data and Methodology

- Mathematical Constraints
  - The mathematical analysis has shown that certain parameter ranges can cause the system to exhibit implausible behaviour (e.g. explode to infinity)
    - Model is simplified, has not accounted for all economic forces
    - Some parameters constrained to regions of meaningful behaviours, e.g. adj. speeds
- Economic Constraints
  - Economically unintuitive parameter values cannot be easily modified to reflect a change in views or the economic environment: usability limitation
    - Economically meaningful ranges from empirical analysis used as soft constraints in the model estimation of some parameter quantities, e.g. profits vs. output
    - Structural assumptions, e.g. IS-LM relationship

Results

- The fit percentage stays almost the same across the estimation and validation datasets
  - No evidence of overfitting
- Forecast correctly identifies q direction change 3/4 of the time; error sources:
  - External events such as 9/11, Asian/Russian Financial Crises 1997 - 1998
  - Structural changes in economy reflected in changing model parameters
- Recalibration of the model based on a moving window of data

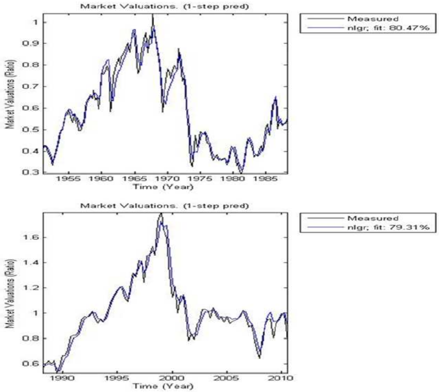
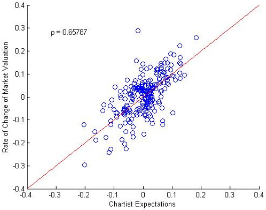
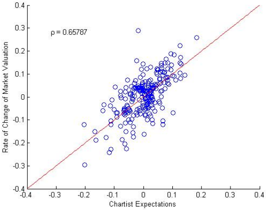


Figure 5.6.: Linearized model fit in estimation dataset (top); in validation dataset (bottom)

Results


ρ = 0.65787
Rate of Change of Market Valuation
Chartist Expectations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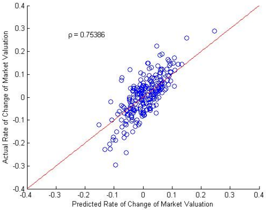
ρ = 0.75386
Actual Rate of Change of Market Valuation
Predicted Rate of Change of Market Valuation

Conclusions

- The model can exhibit oscillatory behaviour, driven by fundamentalists' and chartists' expectations
- The amplitude of oscillations of market valuations is positively related to the valuations' initial state as well as the proportion of chartists in the market
- Dynamical system structure allows for the existence of internal states and the meaningful connection of various economic elements
- Financial markets are a complex interconnected system, affected by a number of global factors, not accounted for in the model

Questions?